Structural Optimization Using Beam Elements

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Techniques for handling beam elements in structural optimization are investigated. The complexities of evaluating the geometrical properties of a section, stress recovery, and design-variable selection are successfully handled by the concept of an element library, use of equivalent stress, and separation of sensitivity variables and design variables. Program implementation is based on the use of an existing finite-element analysis package and a numerical optimizer. The quality of linearization of the behavior constraint functions is also investigated.

Introduction

CONSIDERABLE progress has been made in recent years in efficiency of the structural synthesis process. Of particular importance has been the development of approximation techniques.[1-3] The most common of these has been use of the reciprocal variables in order to produce high-quality approximations to the constraint functions. This technique provides important progress in structural optimization where finite-element structural analysis methods and mathematical programming techniques are combined. Approximation techniques as well as more efficient numerical optimization algorithms have reduced the number of complete analyses to about 10.[4-6]

In this paper the optimization problem takes advantage of high-quality approximations of the behavior constraint functions. First-order Taylor-series expansions are used and the first derivatives of structural responses are always needed. The synthesis procedure is divided into three phases: structural analysis, sensitivity analysis, and optimum design.

In structural analysis the necessry structural responses are obtained; and, in sensitivity analysis the first derivatives of these responses with respect to the most convenient variables, called "sensitivity variables," are obtained. In the optimum design procedure, a mathematical programming problem is formulated using the information provided in the proceeding phases. A key idea is that design variables used in formulating the mathematical programming problem can be selected differently from the sensitivity variables. The quantities obtained in the sensitivity-variable space are transferred into the design-variable space by the Jacobian matrix. The design variables are, in general, selected as the real sizes of structural elements or the reciprocals of these. These three phases can be performed rather independently with respect to selecting parameters in each procedure and developing the computer programs.

In order to treat beam structures, selection of parameters to execute stress analysis, sensitivity analysis, and optimum design is a major concern. In the static structural analysis the variables appearing explicitly in the analysis equations may be the geometrical properties of cross sections. Therefore, it is most convenient to select these as the sensitivity variables. For efficiency the sensitivity variables are selected here with geometrical properties in scaled form, reducing numerical error. The sensitivity analysis is then executed independently of the actual shape of cross sections.

On the other hand, in optimum design for minimum weight, if these geometrical properties are selected as design variables, difficulties arise because the weight is not an explicit function of moments of inertia and the actual values of stress levels are not exactly calculated. This may be overcome to some degree by relating the moment of inertia approximately to the section area.[7,8] But another difficulty still exists: The real sizes of a section must be determined from the optimum-section properties. This introduces another optimization problem that may not have a unique, or even feasible, solution.

Alternatively, the real sizes of a beam cross section can be considered to be the design variables. These are related to the geometrical properties and, therefore, to the sensitivity variables.

Use of reciprocal variables in optimization problems has been well investigated with truss, membrane, and plate structures. Regarding beam structures, in order to obtain a high-quality linearization of functions in the problem, the reciprocal variables of geometrical properties should be used. But, as second best choice, reciprocals of real dimensions of a section can be used to obtain high-quality linearizations. These two approaches are investigated in the present paper with a simple rectangular cross section.

As stated above, the real shape of the cross section is recognized only in the design-optimization phase. The concept of an element library is adopted here, in which typical beam element cross sections are included. The necessary information, such as relations between the geometrical properties and real sizes, and data to calculate maximum stress components in sections, are provided in this library. The biggest advantage of the element library concept is that the mainstream of the synthesis process is not disturbed by complexities lying in individual shapes of sections. The library is accessed with the values of design variables and mainstream control requirements and yields the numerical values of stresses, their derivatives, and the required Jacobian matrices.

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Regarding evaluation of stress levels and formulation of stress constraints, the equivalent stresses (von Mises stress) are calculated at preselected exteme points in the beams. The equivalent stress is limited here to the allowable stress in yielding.

The resultant stresses considered with beam elements are calculated from axial and shear forces, and torsional and bending moments. Corresponding to these forces and moments, up to six geometrical properties are considered. Distributed-element loads and offset among the elements, or a mix with other kinds of elements such as truss and membrane, are not considered in this work.

Several features in program implementation are given here and an existing finite-analysis program is used without modification. The optimizer is also used as a "black box." The concepts of design-variable groups and constraint groups are introduced in order to implement design-variable linking and explicit-constraint deletion.

Sensitivity Variables and Design Variables

An important idea introduced here in dealing with beam elements with various sections is to separate the variables in the sensitivity analysis from those in the optimization process. The actual dimensions or sizes of a beam section are dependent on shape of the cross section. The number of variables to describe a section can be different in different sections and, therefore, their geometrical meanings are also different, while geometrical properties can be common among the elements.

It is most convenient to separate the variables in each phase in structural synthesis as follows: the geometrical properties for structural analysis, their scaled forms for sensitivity analysis, and the real sizes in direct or reciprocal form for optimization.

In sensitivity analysis, the derivatives of structural responses are first obtained with respect to geometrical properties of a section. Recognition of the real shape of a section is not needed at this point. This makes the sensitivity analysis straightforward. Here the variables used in sensitivity analysis are the scaled section properties, called sensitivity variables, and defined with an element by

$$Y=\{A_1\ A_2\ A_3\ \bar{I}_1\ \bar{I}_2\ \bar{I}_3\}^T \tag{1}$$

where

$$\bar{I}_i=\sqrt{I_i}, \qquad i=1,2,3 \tag{2}$$

and A_1, A_2, A_3, and I_1, I_2, I_3 represent axial and effective shear stress areas, torsional, and bending moments of inertia of a beam in three directions (Fig. 1). The use of Eq. (2) is based on dimensionality considerations to improve numerical conditioning.

In order to calculate the derivatives of structural behavior in static analysis the method of pseudoloads is adopted and derivatives of an element-stiffness matrix are necessary. Differentiation may be conducted by

$$\frac{\partial K_e}{\partial Y_i}=\frac{\partial K_e}{\partial A_i}, \qquad i=1,2,3 \tag{3a}$$

$$\frac{\partial K_e}{\partial Y_i}=\frac{\partial K_e}{\partial I_j}\frac{\partial I_j}{\partial \bar{I}_j}=2\sqrt{I_j}\frac{\partial K_e}{\partial I_j}, \qquad i=4,5,6 \quad j=1,2,3 \tag{3b}$$

where K_e stands for an element-stiffness matrix.

The approximate derivatives are easily calculated from the element-stiffness matrix given in Eq. (9).

In the optimum-design phase the real shape of a beam section must be recognized in order to calculate the stresses and formulate the optimization problem. The variables in this phase are real sizes of cross sections denoted by D_i or their reciprocals $1/D_i$. The mathematical programming problem is formulated and solved in the design space, the coordinates of which are called design variables and denoted by

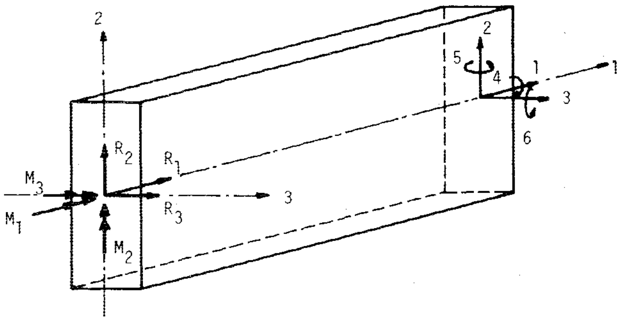


Fig. 1 Beam element.

$$X\equiv\{D_1 D_2 \ldots D_n\}^T \tag{4a}$$

or

$$X\equiv\left\{\frac{1}{D_1}\ \frac{1}{D_2} \cdots \frac{1}{D_n}\right\}^T \tag{4b}$$

The use of reciprocal variables is not critical here since the optimization phase will deal with an explicit approximate problem. However, this has been found to improve the numerical conditioning somewhat and, therefore, is justified. The total number of design variables is denoted by n. The derivatives obtained in the sensitivity analysis are transferred into the design variable space by

$$\left\{\frac{\partial f}{\partial X_i}\right\}=[J_{ij}]\left\{\frac{\partial f}{\partial Y_j}\right\} \tag{5}$$

where $[J_{ij}]$ is the Jacobian matrix defined by

$$J_{ij}=\frac{\partial Y_j}{\partial X_i}, \qquad i=1,\ldots,n \quad j=1,\ldots,m \tag{6}$$

where m stands for number of sensitivity variables. The components of the Jacobian must be obtained for each cross section. The derivatives of geometrical properties with respect to the actual dimensions are straightforward and depend on the actual element being designed. Note that the sensitivity variables of scaled moments of inertia should have the scaling factor with its derivatives as

$$\frac{\partial \bar{I}_j}{\partial X_i}=\frac{1}{2\sqrt{I_j}}\frac{\partial I_j}{\partial X_i} \tag{7}$$

where X_i and I_j are assumed to be of same element.

Since in most cases design variables are selected as the reciprocals of real dimensions, the derivatives are formed by the chain rule as

$$\frac{\partial Y_j}{\partial X_i}=\sum_{k=1}^{n}\frac{\partial Y_j}{\partial D_k}\frac{\partial D_k}{\partial X_i}=-D_i^2\frac{\partial Y_j}{\partial D_i} \tag{8}$$

Element Stiffness Matrix and Its Derivatives

The derivatives of an element-stiffness matrix are obtained only with respect to sensitivity variables, i.e., the scaled geometircal properties. An element-stiffness matrix employed here is a three-dimensional beam with 6 degrees of freedom per node, given by[9]

$$
K_e = \begin{bmatrix}
\frac{EA_1}{L} & & & & & & & & & & & \\
0 & \frac{12EI_3}{L^3(1+\phi_2)} & & & & & & & & & & \\
0 & 0 & \frac{12EI_2}{L^3(1+\phi_3)} & & & & & & & & & \\
0 & 0 & 0 & \frac{GI_1}{L} & & & & & & & & \\
0 & 0 & \frac{-6EI_2}{L^2(1+\phi_3)} & 0 & \frac{(4+\phi_3)EI_2}{L(1+\phi_3)} & & & & \text{Symmetric} & & & \\
0 & \frac{6EI_3}{L^2(1+\phi_2)} & 0 & 0 & 0 & \frac{(4+\phi_2)EI_3}{L(1+\phi_2)} & & & & & & \\
\frac{-EA_1}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA_1}{L} & & & & & \\
0 & \frac{-12EI_3}{L^3(1+\phi_2)} & 0 & 0 & \frac{-6EI_3}{L^2(1+\phi_2)} & 0 & 0 & \frac{12EI_3}{L^3(1+\phi_2)} & & & & \\
0 & 0 & \frac{-12EI_3}{L^3(1+\phi_3)} & 0 & \frac{6EI_2}{L^2(1+\phi_3)} & 0 & 0 & 0 & \frac{12EI_2}{L^3(1+\phi_3)} & & & \\
0 & 0 & 0 & \frac{-GI_1}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GI_1}{L} & & \\
0 & 0 & \frac{-6EI_2}{L^3(1+\phi_3)} & 0 & \frac{(2-\phi_3)EI_2}{L(1+\phi_3)} & 0 & 0 & 0 & \frac{6EI_2}{L^2(1+\phi_3)} & 0 & \frac{(4+\phi_3)EI_2}{L(1+\phi_3)} & \\
0 & \frac{6EI_3}{L^2(1+\phi_2)} & 0 & 0 & 0 & \frac{(2-\phi_2)EI_3}{L(1+\phi_2)} & 0 & \frac{-6EI_3}{L^2(1+\phi_2)} & 0 & 0 & 0 & \frac{(4+\phi_2)EI_3}{L(1+\phi_2)}
\end{bmatrix} \tag{9}
$$

where

$$\phi_2 = \frac{12EI_3}{GA_2L^2} \tag{10a}$$

$$\phi_3 = \frac{12EI_2}{GA_3L^2} \tag{10b}$$

E, G, and L are Young's modulus, shear modulus, and element length, respectively. The deriviates to be obtained are given by Eq. (3) and the derivatives with respect to the geometrical properties

$$\frac{\partial K_e}{\partial A_1}, \quad \frac{\partial K_e}{\partial A_2}, \quad \frac{\partial K_e}{\partial A_3}, \quad \frac{\partial K_e}{\partial I_1}, \quad \frac{\partial K_e}{\partial I_2}, \quad \frac{\partial K_e}{\partial I_3}$$

are calculated by differentiation of Eq. (9).

Stress Constraint and Derivatives

The stress levels are examined at up to four extreme points per section at both ends of an element. This provides sufficient stress-recovery points to identify critical stress in the section under combined stress conditions. The number of stress-recovery points and their locations are fixed in each section. They are shown in Fig. 2 for a rectangular and wide-flange section. The equations that calculate stress components from the forces and moments in the section are written for each type of element and contained in the element library.

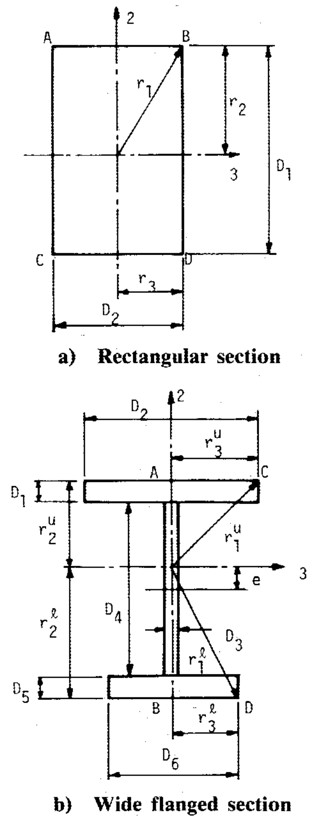


Fig. 2 Dimensions and stress recovery points.

Let σ_{11}, σ_{12}, σ_{13}, and σ_{23} be the stress components of a section. The stress level is evaluated by an equivalent stress

$$\sigma_e = (\sigma_{11}^2 + 3\sigma_{12}^2 + 3\sigma_{13}^2 + 3\sigma_{23}^2)^{1/2} \tag{11}$$

The stress-constraint function is then formulated by

$$g = 1 - \frac{\sigma_e}{\sigma_a} \tag{12}$$

where σ_a is the corresponding allowable stress in yielding. These quantities must be specified in each selected location in the structure. The symbols with no subscripts are assumed to represent all of these quantities. The gradient of the constraint function in the design variable space is given by

$$\nabla^T g \equiv \left\{ \frac{\partial g}{\partial X_1} \frac{\partial g}{\partial X_2} \cdots\cdots \frac{\partial g}{\partial X_n} \right\} \tag{13}$$

where

$$\frac{\partial g}{\partial X_i} = \sum_{j=1}^{m} \frac{\partial g}{\partial Y_j} \frac{\partial Y_j}{\partial X_i}, \qquad i = 1,\ldots,n \tag{14a}$$

or

$$\frac{\partial g}{\partial X_i} = -\frac{1}{\sigma_a} \sum_{j=1}^{m} \frac{\partial \sigma_e}{\partial Y_j} \frac{\partial Y_j}{\partial X_i}, \qquad i = 1,\ldots,n \tag{14b}$$

The term of summation of the right-hand side of Eq. (14b) is the same as in Eq. (5). To calculate this, $\partial\sigma_e/\partial Y_j$ must be obtained. This term may be expanded to

$$\frac{\partial \sigma_e}{\partial Y_j} = \frac{1}{\sigma_e}\left(\sigma_{11} \frac{\partial \sigma_{11}}{\partial y_j} + 3\sigma_{12} \frac{\partial \sigma_{12}}{\partial Y_j} + 3\sigma_{13} \frac{\partial \sigma_{13}}{\partial Y_j} + 3\sigma_{23} \frac{\partial \sigma_{23}}{\partial Y_j} \right) \tag{15}$$

The derivatives of stress components with respect to sensitivity variables are calculated for each different type of cross section. For a rectangular cross section, for example, derivatives of stress components are given by

$$\frac{\partial \sigma_{11}}{\partial Y_j} = -\frac{1}{A_1} \frac{\partial R_1}{\partial Y_j} + \frac{1}{A_1^2} \frac{\partial A_1}{\partial Y_j} + \left(\frac{r_3}{I_2} \frac{\partial M_2}{\partial Y_j} - \frac{M_2 r_3}{I_2^2} \frac{\partial I_2}{\partial Y_j} \right) + \left(\frac{r_2}{I_3} \frac{\partial M_3}{\partial Y_j} - \frac{M_3 r_2}{I_3^2} \frac{\partial I_3}{\partial Y_j} \right) \tag{16a}$$

$$\frac{\partial \sigma_{12}}{\partial Y_j} = -\frac{1}{A_2} \frac{\partial R_2}{\partial Y_j} + \frac{R_2}{A_2^2} \frac{\partial A_2}{\partial Y_j} \tag{16b}$$

$$\frac{\partial \sigma_{13}}{\partial Y_j} = -\frac{1}{A_3} \frac{\partial R_3}{\partial Y_j} + \frac{R_3}{A_3^2} \frac{\partial A3}{\partial Y_j} \tag{16c}$$

$$\frac{\partial \sigma_{23}}{\partial Y_j} = -\frac{r_1}{I_1} \frac{\partial M_1}{\partial Y_j} + \frac{M_1 r_1}{I_1^2} \frac{\partial I_1}{\partial Y_j} \tag{16d}$$

where the forces and moments in direction 1, 2, and 3 are denoted by R_1, R_2, R_3, and M_1, M_2, M_3, respectively. A_1, A_2, A_3 and I_1, I_2, I_3 are geometrical properties as stated in the section on Sensitivity Variables and Design Variables and stress calculation points r_1, r_2, and r_3 are shown in Fig. 2.

In these equations the derivatives of the stress resultants $R_1 - M_3$ with respect to the sensitivity variables Y_j are the products of the sensitivity analysis. The derivatives of the geometrical properties $A_1 - I_3$ are zeros if Y_j does not represent the element under consideration and, if it does, unity for the derivatives of A and $2\sqrt{I_k}$ for I, where $Y_j = \sqrt{I_k}$. Equations of stress-component derivatives with a wide-flange beam are similarly calculated using geometry of the beam.

Optimization Problem and Program Implementation

The minimum weight-design problem is formulated subject to stress and displacement constraints. Additionally, geometric constraints are used to impose realistic sizing constraints such as limits on web height-to-thickness ratio. Having created the approximate problem at each optimization stage by generating first-order Taylor-series expansions with respect to design variables, the optimization task becomes: Find the change in design variables δx that will

minimize

$$F(X + \delta X) \tag{17a}$$

subject to

$$g_j(X) + \nabla g_j \cdot \delta X \leq 0, \qquad j = 1,\ldots,p \tag{17b}$$

$$-C_i \leq \delta X_i \leq C_i, \qquad i = 1,\ldots,n \tag{17c}$$

$$X_i^\ell \leq X_i + \delta X_i \leq X_i^u, \qquad i = 1,\ldots,n \tag{17d}$$

where C_i is used to limit design changes to the region of applicability of the linearizations, and X_i^ℓ and X_i^u are side constraints imposed directly on the design variables. The move limits C_i are considered necessary in any approximation method. In the examples to follow, the move limits were critical only in the early design stages, indicating that the approximations used here are, indeed, of high quality. If the problem is formulated in the reciprocal variable space, the mathematical programming problem consists of a nonlinear objective function with linear constraints.

Several important features are introduced here in implementing the computer program. A basic philosophy of program design is to use an existing finite-element analysis program with little or no modifications as well as the numerical optimization program. In this case, the method is implemented using the SAP4[10] and ADS[11] programs.

The design-variable linking is implemented by the concept of design-variable groups. The structural finite elements of same type (truss, beam, and membrane) that share common design variables are grouped together. Therefore, the design variables or sensitivity variables are representations of the group rather than individual elements. The variable linking in general form must be added to the geometical constraints in restricted form in which the linearization must not be disturbed.

The explicit-constraint deletion is also given by the concept of the contraint group, in which the displacement constraints at selected points in the structure with the same component and limit are also grouped in order to avoid redundant constraints.

A capability of the active sensitivity-variable selection allows use of only significant geometrical properties of a beam element depending on the type of problem. The sensitivity coefficients are calculated with respect to only the active sensitivity variables.

Numerical Examples

Quality of Linearizations

A simple cantilever beam of rectangular section in bending is solved to investigate the quality of linearization of a stress

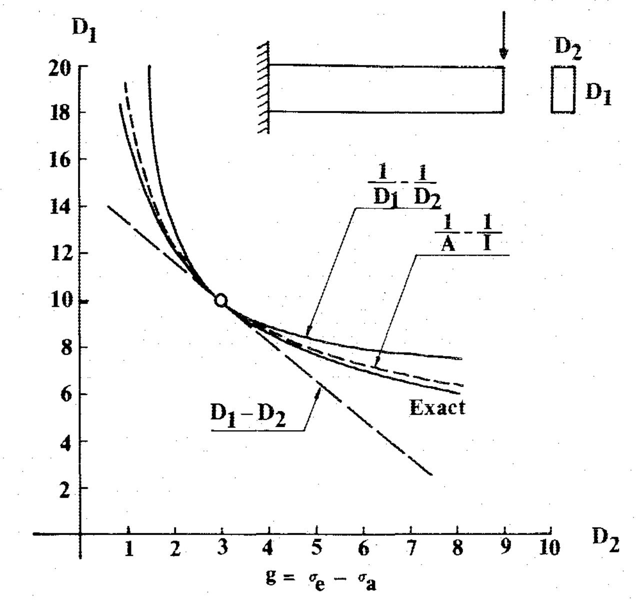


Fig. 3 Quality of approximations.

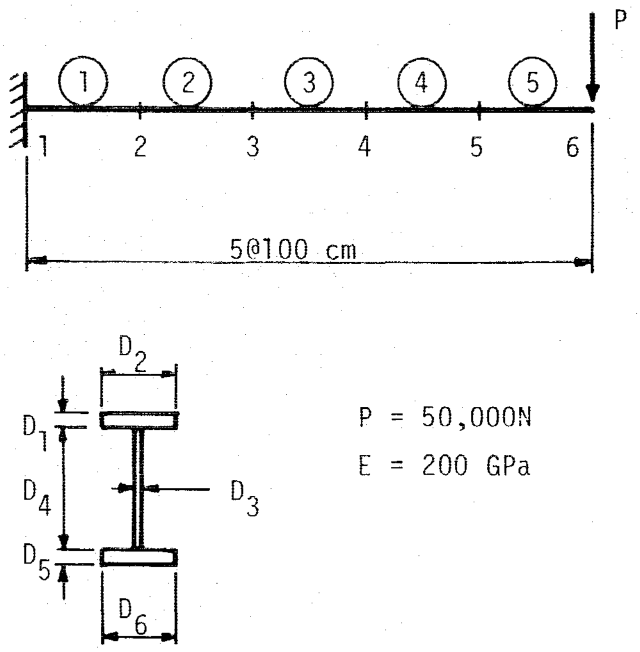


Fig. 4 Cantilevered beam.

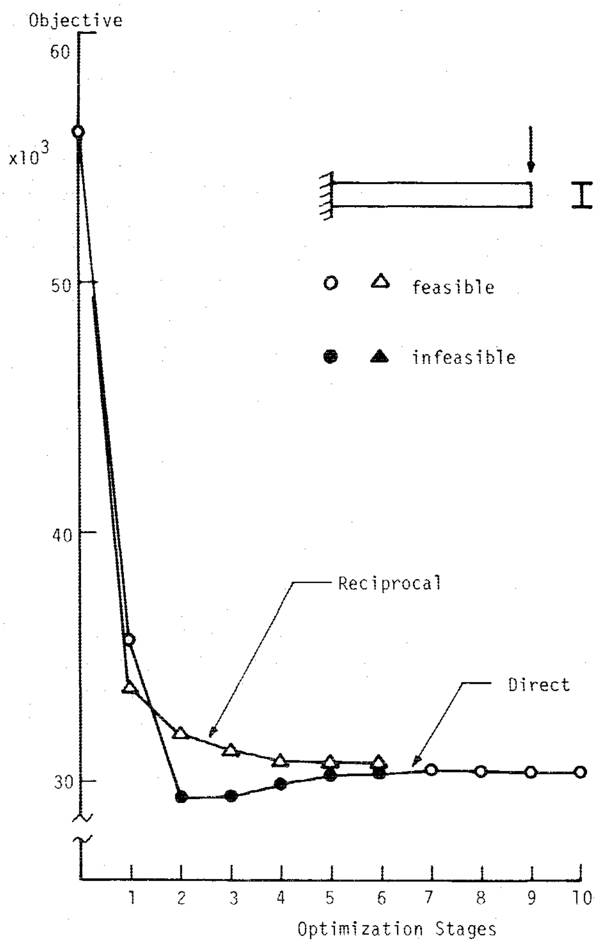


Fig. 5 Design history for cantilevered beam.

constraint. The results are shown in Fig. 3 where the linearizations in different coordinates are plotted in the real-size coordinate space. The equivalent stress is calculated at the upper limit fiber of the beam at the fixed end and the normal and shear stresses are considered. The linearization with respect to the reciprocals of real dimensions $1/D_1 - 1/D_2$ gives a conservative approximation. Comparing with the selection of direct variables $D_1 - D_2$, the advantage is obvious. The linearization with respect to the reciprocals of the section properties is close to the exact solution (that includes shear deformation).

Simple Cantilevers

A simple cantilever beam with wide-flange cross section as shown in Fig. 4 is to be designed for minimum material volume. Five beam elements are used and a total of 30 design variables are involved in the problem. Because of the symmetry in the problem, however, the upper and lower flanges must be the same. The selected sensitivity variable of an element is only the moment of inertia around the axis 3. The cantilever beam is designed under the static load and linear elastic stress and displacement constraints. The displacement of the free end must be less than 4.0 cm in the vertical direction. The stresses are evaluated at the top fiber of the upper flanges where the shear-stress components are considered as well as the bending stresses. The allowable stress used here is 14,000 N/cm^2 for all elements. The geometrical constraints are also imposed so that the plate thickness-width ratio is less than 20.0 for flange plates and 100.0 for web plates. The minimum gage limits are 3.0 for all flanges and 6.0 for all webs. There are no upper bounds imposed for these dimensions.

The optimum designs are obtained with two approximation methods: one with the direct-sensitivity variables and the other with the reciprocal-sensitivity variables. The linearizations with reciprocal variables are assumed to be conservative and non-conservative with direct variables. Thus, move limits of 25 and 50% are used for the direct and reciprocal linearizations, respectively. Comparison of these two methods is exhibited with the computation history of the objectives in Fig. 5. The numerical values of objective function and design variables are listed in Tables 1 and 2 for two approximate methods. The values of constraint functions are listed in Tables 3 and 4.

A significant feature may be observed from the reciprocal variables example. At the first stage a fully stressed design is obtained with respect to linearized stress constraints while 6–23% of positive margin of exact constraints are still maintained. For the following design stages the linearized problem yields the fully stressed designs; and, the positive margin of the exact constraints are gradually reduced at 2–11, 1–7, −2–4, −1–3, and, finally, −1–3% at the sixth design stage.

Table 1 Objective and design variables of a cantilever beam (direct variables)[a]

		Optimization stage					
		0	1	2	3	......	10
Objective		56,000	35,706	29,362	29,387	......	30,414
Design var.							
El.#1	D_1	3.00	2.25	2.81	3.12	......	3.48
	D_2	10.00	7.50	5.63	5.00	......	5.04
	D_3	1.00	0.75	0.69	0.70	......	0.71
	D_4	60.00	68.16	69.30	70.40	......	70.67
El.#2	D_1	3.00	2.25	2.20	2.54	......	2.73
	D_2	10.00	7.50	5.63	5.00	......	5.01
	D_3	1.00	0.75	0.68	0.68	......	0.68
	D_4	60.00	63.19	68.35	68.32	......	68.46
El.#3	D_1	3.00	2.25	1.69	2.02	......	2.19
	D_2	10.00	7.50	5.70	5.00	......	5.01
	D_3	1.00	0.75	0.63	0.63	......	0.63
	D_4	60.00	55.03	62.90	62.78	......	62.89
El.#4	D_1	2.00	1.50	1.13	1.06	......	1.24
	D_2	10.00	7.50	5.63	5.00	......	5.01
	D_3	1.00	0.75	0.60	0.60	......	0.60
	D_4	60.00	49.69	56.50	59.79	......	60.07
El.#5	D_1	2.00	1.50	1.13	0.84	......	0.84
	D_2	10.00	7.50	5.63	5.00	......	5.00
	D_3	1.00	0.75	0.60	0.60	......	0.60
	D_4	60.00	45.00	40.85	43.63	......	43.88

[a]$D_5 = D_1$, $D_6 = D_2$ for all elements.

Table 2 Objective and design variables of a cantilever beam (reciprocal variables)[a]

		Optimization stage					
		0	1	2	3	......	6
Objective		56,000	33,784	31,980	31,325	......	30,854
Design var.							
El.#1	D_1	3.00	2.65	2.64	2.64	......	2.65
	D_2	10.00	7.56	6.87	6.76	......	5.73
	D_3	1.00	0.69	0.71	0.71	......	0.74
	D_4	60.00	69.72	70.86	70.85	......	74.53
El.#2	D_1	3.00	2.56	2.52	2.50	......	2.50
	D_2	10.00	7.24	6.44	6.16	......	5.48
	D_3	1.00	0.67	0.67	0.67	......	0.69
	D_4	60.00	66.70	67.00	66.88	......	68.70
El.#3	D_1	3.00	2.21	2.15	2.14	......	2.13
	D_2	10.00	8.10	5.79	6.23	......	5.63
	D_3	1.00	0.60	0.60	0.60	......	0.60
	D_4	60.00	60.00	60.20	60.30	......	61.46
El.#4	D_1	2.00	1.60	1.57	1.56	......	1.51
	D_2	10.00	5.87	5.00	5.00	......	5.00
	D_3	1.00	0.62	0.61	0.60	......	0.60
	D_4	60.00	61.38	60.33	58.55	......	57.31
El.#5	D_1	2.00	1.43	1.41	1.41	......	1.37
	D_2	10.00	5.00	5.00	5.00	......	5.00
	D_3	1.00	0.68	0.67	0.67	......	0.64
	D_4	60.00	37.25	36.36	35.76	......	35.86

[a]$D_5 = D_1$, $D_6 = D_2$ for all elements.

The designs obtained with reciprocal variables in this example always remain in the feasible region of the design space. This is an important and useful feature in practical design optimization.

In the direct variable approximations example, the design at the second through sixth stages are infeasible. This may be expected with the characteristics of the linearization of constraint functions. There is a tradeoff between move limit and computation time.

This example is presented in order to compare the two approximations techniques. The optimization process was terminated with a rough criteria of the minimum design improvement.

Gridwork

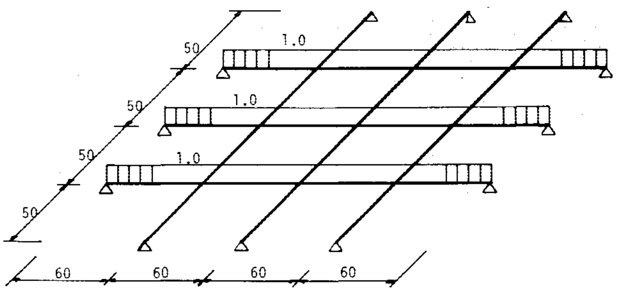
A grid work shown in Fig. 6a is to be designed here for minimum material volume. The finite-element model for a quarter of the structure is shown in Fig. 6b. Wide-flange sec-

Table 3 Constraint values of a cantilevered beam design (direct variables)

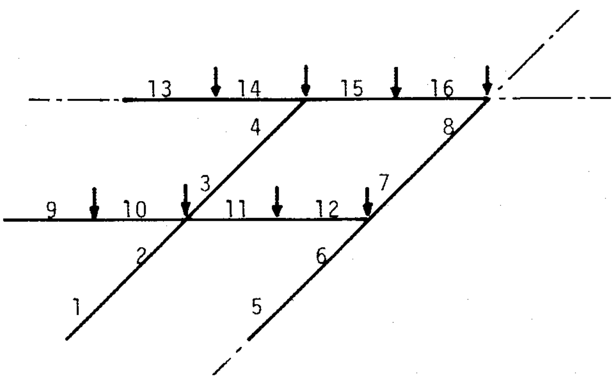
	Optimization stage					
	0	1	2	3	......	10
Displacement pt. 6	−0.713	−0.554	−5.111	−0.527	......	−0.559
Stress #1	−0.234	0.059	0.116	0.091	......	0.009
#2	−0.385	−0.061	0.070	0.054	......	0.004
#3	−0.534	−0.152	0.083	0.060	......	0.003
#4	−0.584	−0.145	0.091	0.092	......	0.002
#5	−0.776	−0.491	−0.110	−0.001	......	−0.009
Web plate ratio #1	−40.00	−6.84	0.0	0.0	......	−0.02
#2	−40.00	−11.80	0.0	0.0	......	0.0
#3	−40.00	−20.00	0.0	0.0	......	−0.01
#4	−40.00	−25.30	−3.50	−0.21	......	−0.05
#5	−40.00	−30.00	−19.20	−16.40	......	−16.20

Table 4 Constraint values of a cantilevered beam design (reciprocal variables)

		Optimization stage			
		0	1	2	3
Displacement pt. 6	−0.713	−0.610	−0.578	−0.562	−0.559
Stress #1	−0.234	−0.062	−0.021	0.010	0.012
#2	−0.385	−0.148	−0.071	−0.035	−0.005
#3	−0.534	−0.222	−0.109	−0.049	−0.016
#4	−0.584	−0.228	−0.117	−0.075	−0.027
#5	−0.776	−0.107	−0.063	−0.041	−0.016
Web plate ratio #1	−0.667	0.010	0.003	0.0	0.006
#2	−0.667	0.002	0.0	0.0	−0.002
#3	−0.667	0.0	0.0	0.006	−0.001
#4	−0.667	−0.011	−0.008	−0.054	−0.078
#5	−0.667	−1.220	−1.260	−1.300	−1.240



a) **Dimensions and loads**



b) **Finite-element model**

Fig. 6 Grid work.

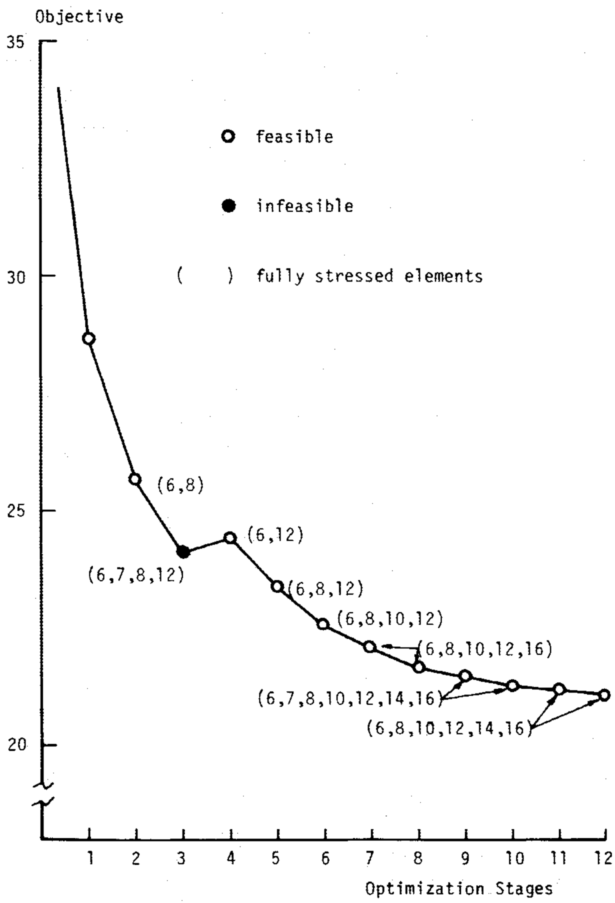


Fig. 7 Design history for grid work.

tions are used for all elements. The design variables are assumed to be identical for elements 1 through 16. Each design-variable group consists of four independent variables since the upper and lower flanges must be identical. The problem consists of a total of 32 independent design variables.

Table 5 Objective and design variables of grid work design (reciprocal variables)

		Optimization stage						
		0	1	2	3	......	12	Ref. 12
Objective		37,300	28,652	25,678	24,074	......	21,047	23,190
Design var.								
El.#1	D_1	0.250	0.250	0.250	0.250	......	0.250	
	D_2	5.000	4.867	4.739	4.619	......	3.734	
	D_3	0.250	0.250	0.250	0.250	......	0.249	
	D_4	20.000	17.000	16.000	15.550	......	10.240	
	A	7.500	6.709	6.370	6.197	......	4.417	5.10
El.#2	D_1	0.250	0.250	0.250	0.250	......	0.250	
	D_2	5.000	4.835	4.679	4.533	......	3.540	
	D_3	0.250	0.250	0.250	0.250	......	0.250	
	D_4	20.000	12.290	9.430	7.657	......	7.403	
	A	7.500	5.490	4.697	4.181	......	3.613	5.10
El.#3	D_1	1.000	0.993	0.985	0.978	......	0.919	
	D_2	20.000	12.310	8.892	6.960	......	2.364	
	D_3	1.000	0.996	0.992	0.988	......	0.942	
	D_4	40.000	32.160	34.860	37.840	......	53.080	
	A	80.000	56.480	52.098	51.000	......	54.346	47.50
El.#4	D_1	1.000	0.993	0.986	0.980	......	0.924	
	D_2	20.000	12.944	9.564	7.584	......	2.652	
	D_3	1.000	0.996	0.990	0.986	......	0.936	
	D_4	40.000	36.750	39.070	41.970	......	60.030	
	A	80.000	62.310	57.540	56.248	......	61.088	55.80
El.#5	D_1	1.000	0.994	0.988	0.982	......	0.929	
	D_2	5.000	4.376	3.880	3.486	......	1.797	
	D_3	0.250	0.250	0.250	0.250	......	0.248	
	D_4	40.000	24.870	20.090	17.750	......	20.490	
	A	20.000	14.917	12.689	11.284	......	8.420	13.00
El.#6	D_1	1.000	0.994	0.988	0.982	......	0.931	
	D_2	5.000	4.383	3.891	3.499	......	1.817	
	D_3	0.250	0.250	0.250	0.250	......	0.248	
	D_4	40.000	26.050	20.610	17.870	......	29.180	
	A	20.000	15.226	12.841	11.340	......	10.622	17.00
El.#7	D_1	1.000	0.994	0.987	0.980	......	0.923	
	D_2	10.000	8.644	7.598	6.780	......	3.386	
	D_3	0.500	0.500	0.500	0.500	......	0.496	
	D_4	40.000	24.030	20.130	18.330	......	13.450	
	A	40.000	29.200	25.064	22.454	......	12.922	13.60
El.#8	D_1	1.000	0.994	0.988	0.983	......	0.930	
	D_2	10.000	8.764	7.794	7.016	......	3.618	
	D_3	0.500	0.500	0.500	0.500	......	0.496	
	D_4	40.000	25.720	25.800	25.190	......	16.440	
	A	40.000	30.282	28.300	26.388	......	14.884	16.60

Active sensitivity variables are selected as the bending moments of all elements. The behavior constraints considered are static stress constraints. With the shear and bending stress components the maximum equivalent stresses of the cross sections are evaluated at both ends of an element. The constraints on web height-thickness ratio are taken as 1000. The minimum gage limits are also imposed with 0.1 for flanges and 0.5 for webs. No upper limits on design variables are assumed. The method of linearization of the constraint functions are applied with the reciprocal variables of the real dimensions of sections.

The optimization history is shown in Fig. 7 with the values of objective at each design stage. At the third stage a 5% stress-constraint violation is observed at element 8. The numbers in each stage in Fig. 7 indicate the element numbers of fully stressed members.

The numerical values of the objective and design variables are listed in Table 5. The results of Ref. 12 are shown in this table, which imposed the minimum limits on cross-sectional areas of the beams of 5.0. The method presented here using the real dimensions of the beam is successfully conformed. However, there are insignificant design-variable changes observed among plate thickness. To optimize these variables, additional design stages may be necessary and the optimization becomes inefficient. It must be noted here that selection of the significant design variables depending on the type of problem is important. In this example, the bending moment of inertia is the only significant geometrical property and the number of possible design variables of a section should be reduced for efficient optimization.

Summary

One of the purposes of this work is to study how to handle beam elements with the real cross-sectional dimensions. With this approach, the optimization flow is well arranged and independent of the various shapes of cross sections. The element library supports the mainstream of optimization as the utilities and provides the numerical data including geometrical properties, stresses, their derivatives, and the Jacobian relating sensitivity variables to design variables. The numerical data transferred into the mainstream are common among the various shapes of sections. With this idea the numerical-optimization problem is successfully executed with real sizes as design variables.

The second purpose is to seek high-quality linearizations of the constraint functions. The use of the reciprocal form of real sizing variables gave good results, especially when the approximations are conservative. Two numerical examples with wide-flange beams were give to demonstrate the method. It is indicated, however, that selection of the significant variables is necessary when the active section properties are only one or two in the problem.

By proper selection of sensitivity variables, it has been shown possible to design frame structures using only 10-12 detailed structural analyses. At the same time, the design task itself is formulated in terms of the physical dimensions familiar to the designer. The key concept here is that the design and sensitivity variables are related by the Jacobian, which is dependent on the particular element type. Because the Jacobian is not constant, the constraints in the approximate-optimization stage are nonlinear, but are easily and effectively evaluated. Thus, linearizations are performed at the point needed to provide high-quality approximations to the response quantities while still retaining the essential nonlinearities of the design task.

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